(1) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} \xrightarrow{\text{definujeme}} f(z) = \sum_{n=1}^{\infty} \frac{z^{n+2}}{n(n+2)}, \quad |z| < 1$

↑ poloměr konvergence

Potom pro $|z|<1$ platí

$$f'(z) = \sum_{n=1}^{\infty} \frac{z^{n+1}}{n} = z \cdot \sum_{n=1}^{\infty} \frac{z^n}{n} \overset{(*)}{=} -z \log(1-z) \overset{(*)}{\Rightarrow} f(z) \overset{c}{=} -\log(1-z) \cdot \frac{1-z^2}{2} + \frac{z}{2} + z^2, \quad f(0) = 0 \Rightarrow C = 0$$

(*) pro $|z|<1$ $\left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right)' = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \rightarrow \sum_{n=1}^{\infty} \frac{z^n}{n} \overset{c}{=} -\log(\underset{z>1}{\uparrow}1-z)$, dosazením $z=0$ dostaneme $C=0$ a $\sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z)$

(*) $$\int -z\log(1-z)\,dz = \left| \begin{matrix} u = -\log(1-z) & u' = \frac{1}{1-z} \\ v' = z & v = \frac{z^2}{2} \end{matrix} \right| = -\log(1-z) \cdot \frac{z^2}{2} - \frac{1}{2}\int \frac{z^2}{1-z}\,dz = -\log(1-z)\frac{z^2}{2} - \frac{1}{2}\int \frac{z^2-1}{1-z}dz - \frac{1}{2}\int \frac{1}{1-z}dz$$

$$= -\log(1-z)\frac{z^2}{2} + \frac{1}{2}\int 1+z\,dz + \frac{1}{2}\log(1-z) \overset{c}{=} -\log(1-z) \cdot \frac{1-z^2}{2} + \frac{z}{2} + \frac{z^2}{4}$$

$$f(z) = -\log(1-z) \cdot \frac{1-z^2}{2} + \frac{z}{2} + \frac{z^2}{4}$$

Dále $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} < \infty$ (srovnáním s $\frac{1}{n^2}$)

Podle Abelovy věty
$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{z\to 1-} f(z) = \lim_{z\to 1-} \left(\underbrace{-\log(1-z) \cdot \frac{1-z^2}{2}}_{\text{použijeme } \lim_{x\to 0+} x\log x = 1} + \frac{z}{2} + \frac{z^2}{4} \right) = 0 + \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$$

Alternativně pomocí rozkladu na parciální zlomky $\left(\frac{1}{n(n+2)} = \frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2}\right)$

$$f(z) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{z^{n+2}}{n} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{z^{n+2}}{n+2} = \frac{z^2}{2}\sum_{n=1}^{\infty} \frac{z^n}{n} - \frac{1}{2}\underbrace{\sum_{n=3}^{\infty} \frac{z^n}{n}}_{\sum_{n=1}^{\infty} \frac{z^n}{n} - z - \frac{z^2}{2}} = -\frac{z^2}{2}\log(1-z) + \frac{1}{2}\log(1-z) + \frac{z}{2} + \frac{z^2}{4}$$

Ještě jiná alternativně (bez použití teorie mocninných řad)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)\right) = \frac{1}{2}\lim_{N\to\infty} \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2} \cdot \lim_{N\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \left(\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+2}\right)\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2} \cdot \left(1 + \frac{1}{2}\right) = \boxed{\frac{3}{4}}$$